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LETTER TO THE EDITOR

The quantum Knizhnik–Zamolodchikov equation, generalized Razumov–Stroganov sum rules and extended Joseph polynomials

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Abstract

We prove higher rank analogues of the Razumov–Stroganov sum rule for the ground state of the $O(1)$ loop model on a semi-infinite cylinder: we show that a weighted sum of components of the ground state of the A_{k-1} IRF model yields integers that generalize the numbers of alternating sign matrices. This is done by constructing minimal polynomial solutions of the level 1 $U_q(\widehat{\mathfrak{sl}(k)})$ quantum Knizhnik–Zamolodchikov equations, which may also be interpreted as quantum incompressible q-deformations of quantum Hall effect wavefunctions at filling fraction $\nu = k$. In addition to the generalized Razumov–Stroganov point $q = -e^{i\pi/k+1}$, another combinatorially interesting point is reached in the rational limit $q \to -1$, where we identify the solution with extended Joseph polynomials associated with the geometry of upper triangular matrices with vanishing kth power.

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1. Introduction

The present work stems from the recent activity around the so-called Razumov–Stroganov conjecture [1, 2], which relates the ground-state vector of the $O(1)$ loop model on a semi-infinite cylinder of perimeter $2n$ to the numbers of configurations of the six–vertex (6V) model on a square grid of size $n \times n$, with domain wall boundary conditions (DWBC). In [3], we were able to prove a weaker version of the conjecture, identifying the *total number* of configurations, also equal to the celebrated number of $n \times n$ alternating sign matrices, to the *sum* of entries of the ground-state vector, thus establishing the Razumov–Stroganov sum rule. The proof is more general and actually identifies the ground state Ψ of the fully inhomogeneous version of the loop model with the so-called Izergin–Korepin determinant, i.e. the inhomogeneous partition function of the 6V model with DWBC. It makes extensive use of the integrability of the models. We adapted this proof to the case of the $O(1)$ crossing loop model [4], which is based on a rational (as opposed to trigonometric) solution of the Yang–Baxter equation, and established a direct relation to the geometry of certain schemes of matrices, further developed in [5].

More recently, Pasquier [6] constructed a polynomial representation of the affine Hecke algebra, allowing him to recover the aforementioned sum rule. His method can be reformulated as finding a polynomial solution Ψ to the quantum Knizhnik–Zamolodchikov (qKZ) equation for $U_q(\widehat{\mathfrak{sl}(2)})$ at level 1. At $q = -\mathrm{e}^{\mathrm{i}\pi/3}$, it coincides with the Razumov–Stroganov ground state, but it is defined for arbitrary q as well, although it has no direct interpretation as the ground state of some loop model. We note here that for $q \to -1$, when the trigonometric solution of the Yang–Baxter equation degenerates into a rational one, the entries of Ψ tend to non-negative integers (different from the Razumov–Stroganov case). As we shall see below, these are the degrees of the components of the scheme of upper triangular complex $2n \times 2n$ matrices with square zero.

In this letter, we also address the case of higher rank algebras, i.e. the so-called A_{k-1} vertex/height models. The natural rank k counterpart of the $O(1)$ loop model is constructed by using a path representation for the Hecke algebra quotient associated with $U_q(\mathfrak{sl}(k))$ (we do not use here the more traditional spin chain representation, see e.g. the somewhat related work [7]; though we mention it a few times in the following). We then work out the polynomial solution Ψ to the $U_q(\widehat{\mathfrak{sl}(k)})q$KZ equation at level 1, which, at the particular point $q = -\mathrm{e}^{\frac{\mathrm{i}\pi}{k+1}}$, turns out to be exactly the ground state of a half-cylinder A_{k-1} IRF model, with a transfer matrix acting naturally in the path representation. At this generalized 'Razumov–Stroganov' point, we establish a new sum rule for the entries of Ψ. In particular, we find natural generalizations of the number of alternating sign matrices for arbitrary k. What exactly is counted by these numbers still eludes us. We finally investigate the rational $q \to -1$ limit within the framework of equivariant cohomology in the space of upper triangular matrices, relating it to schemes of nilpotent matrices of order k.

2. A_{k-1} R-matrix and path representation

The standard abstract trigonometric solution of the Yang–Baxter equation reads

$$\check{R}_i(z, w) = \frac{q^{-1}z - qw}{q^{-1}w - qz} + \frac{z - w}{q^{-1}w - qz} e_i, \tag{2.1}$$

where $e_i, i = 1, 2, \ldots, N-1$, are the generators of the Hecke algebra $H_N(\tau)$, with the relations

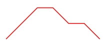
$$e_i e_j = e_j e_i, \quad |i-j| > 1, \qquad e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}, \qquad e_i^2 = \tau e_i, \tag{2.2}$$

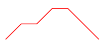
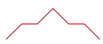
with the parametrization $\tau = -(q + q^{-1})$.

In this letter, we restrict e_i to be the generators of the quotient of the Hecke algebra related to $U_q(\mathfrak{sl}(k))$ in the fundamental representation and denoted by $H_N^{(k)}(\tau)$; it is obtained by imposing extra relations, namely the vanishing of the q-symmetrizers of order k, $Y_k(e_i, e_{i+1}, \ldots, e_{i+k-1})$, defined recursively by $Y_1(e_i) = e_i$ and $Y_{m+1}(e_i, \ldots, e_{i+m}) = Y_m(e_i, \ldots, e_{i+m-1})(e_{i+k} - \mu_m) Y_m(e_i, \ldots, e_{i+m-1})$, with $\mu_m = U_{m-1}(\tau)/U_m(\tau)$, U_m are the

Chebyshev polynomials of the second kind $U_m(2\cos\theta) = \sin\big((m+1)\theta\big)/\sin(\theta)$. For $k = 2$, the quotient $H_N^{(2)}(\tau)$ is nothing but the Temperley–Lieb algebra $\mathrm{TL}_N(\tau)$.

Representations of the algebra $H_N^{(k)}(\tau)$ have been extensively used to construct lattice-integrable vertex models based on A_{k-1}. Here, we consider the so-called path representation, naturally leading to IRF models, and restrict ourselves to the case where $N = nk$. States are indexed by closed paths from and to the origin on the (link-oriented) Weyl chamber of $SU(k)$, allowed to make steps $u_1, u_2, \ldots, u_k$, where $u_1 = \omega_1, u_2 = \omega_2 - \omega_1, \ldots, u_k = -\omega_{k-1}$ in terms of the fundamental weights ω_i. These paths visit only points $\lambda = \sum_{1\leqslant i\leqslant k-1} \lambda_i\omega_i$ with all $\lambda_i \geqslant 0$. It is useful to represent them via their sequence of steps, substituting $u_i \to i$ for simplicity. For instance, the path of length nk closest to the origin is $(12\ldots k)^n$, namely n repetitions of the sequence $1, 2, \ldots, k$, and we denote it by π_{f}. Likewise, the path farthest from the origin is $(1)^n(2)^n\ldots(k)^n$, and we denote it by π_0. A useful notation consists in representing each step j by a unit segment forming an angle of $\frac{\pi(k+2-2j)}{2(k+1)}$ with the horizontal direction: each π becomes a broken line touching the x axis at its ends and staying above it. There are exactly $(kn)!\prod_{0\leqslant j\leqslant k-1} j!/(n+j)!$ such paths. For illustration, for $k = 3, n = 2$ we have the following five A_2 paths of length 6:

$\pi_0 = (112233) \quad\to$  $\qquad \pi_1 = (112323) \quad\to$ 

$\pi_2 = (121233) \quad\to$  $\qquad \pi_3 = (121323) \quad\to$  $\qquad$ (2.3)

$\pi_{\mathrm{f}} = (123123) \quad\to$ 

shown in step and broken-line representations.

Associating a vector $|\pi\rangle \equiv |\pi_1\pi_2\ldots\pi_N\rangle$ with each path, the path representation satisfies the following properties:

(P1) $e_i|\pi\rangle = \tau|\pi\rangle$ if $\pi_i < \pi_{i+1}$ (π locally convex);

(P2) $e_i|\pi\rangle = \sum_{\pi'} C_{i,\pi,\pi'}|\pi'\rangle$ if $\pi_i \geqslant \pi_{i+1}$ (π locally flat or concave), for some $C_{i,\pi,\pi'} \in \{0, 1\}$,

and will be discussed in detail elsewhere. Let us however mention two more properties of crucial importance in the following:

(P3) if $C_{i,\pi,\pi'} = 1$, then π' is locally convex between steps i and $i+1$, namely $\pi'_i < \pi'_{i+1}$;

(P4) if $C_{i,\pi,\pi'} = 1$, then either $(\pi_i, \pi_{i+1}) = (\pi'_{i+1}, \pi'_i)$ and $\pi_m = \pi'_m$ for all $m \neq i, i+1$, i.e. π' exceeds π by a unit lozenge in the broken-line representation, or $\pi' \subset \pi$, namely the broken-line representation of π' lies below that of π.

For illustration, when $k = 3, n = 2$ and $i = 1$, we have $e_1|\pi_0\rangle = |\pi_{\mathrm{f}}\rangle + |\pi_2\rangle$ and $e_1|\pi_1\rangle = |\pi_3\rangle$ while $(e_1 - \tau)$ annihilates $|\pi_2\rangle, |\pi_3\rangle, |\pi_{\mathrm{f}}\rangle$, for the paths of equation (2.3).

Let us finally mention the rotation σ acting on paths as follows. Given a path π, we record its last passages on the walls of the Weyl chamber, namely at points with $\lambda_m = 0$, for $m = 1, 2, \ldots, k-1$. The steps taken from those points are of the type $1, 2, \ldots, k-1$, respectively. The rotated path $\sigma\pi$ is obtained by first 'rotating' these steps, namely transforming them into steps of types $2, 3, \ldots, k$, respectively, while all the other steps are unchanged, then deleting the last step $\pi_N = k$, and finally adding a first step 1. With this definition, we have the rotational invariance: $C_{i+1,\sigma\pi,\sigma\pi'} = C_{i,\pi,\pi'}$, for $i = 1, 2, \ldots, N-2$, which may be recast into $\sigma e_i = e_{i+1}\sigma$, and thus allows for defining an extra generator $e_N = \sigma e_{N-1}\sigma^{-1} = \sigma^{-1}e_1\sigma$. As an example, the paths of equation (2.3) form two cycles under the rotation σ, namely $\pi_0 \to \pi_3 \to \pi_0$ and $\pi_1 \to \pi_2 \to \pi_{\mathrm{f}} \to \pi_1$.

3. The quantum Knizhnik–Zamolodchikov equation

The quantum Knizhnik–Zamolodchikov (qKZ) equation [8] is a linear difference equation satisfied by matrix elements of intertwiners of highest weight representations of affine quantum groups (here, $U_q(\widehat{\mathfrak{sl}(k)})$). It can be reformulated as the following equivalent set of conditions:

$$\tau_i \Psi(z_1, \ldots, z_N) = \check{R}_i(z_{i+1}, z_i)\Psi(z_1, \ldots, z_N), \qquad i = 1, 2, \ldots, N-1, \tag{3.1a}$$

$$\Psi(z_2, z_3, \ldots, z_N, sz_1) = c\sigma^{-1}\Psi(z_1, \ldots, z_N), \tag{3.1b}$$

where $\Psi = \sum_\pi \Psi_\pi |\pi\rangle$ is a vector in the path representation defined above. In (3.1a), τ_i acts on functions of zs by interchanging z_i and z_{i+1}. In (3.1b), σ is the rotation operator: $(\sigma^{-1}\Psi)_\pi = \Psi_{\sigma\pi}$; c is an irrelevant constant which can be absorbed by homogeneity; s determines the *level* l of the qKZ equation via $s = q^{2(k+l)}$, where k plays here the role of dual Coxeter number. Note that instead of imposing equation (3.1b), one could consider only equations (3.1a) but for any i, with the implicit shifted periodic boundary conditions $z_{i+N} = sz_i$, and the various $\check{R}_i$ related to each other by conjugation by σ.

We introduce the operators t_i, acting locally on the variables z_i and z_{i+1} of functions f of $z_1, z_2, \ldots, z_N$ via

$$t_i f = (qz_i - q^{-1}z_{i+1})\partial_i f, \tag{3.2}$$

where $\partial_i = \frac{1}{z_{i+1}-z_i}(\tau_i - 1)$ is the divided difference operator[3]. Equation (3.1a) is equivalent to $t_i\Psi = (e_i - \tau)\Psi$, and indeed one can check that $t_i + \tau$, or equivalently $-t_i$, satisfy by construction the Hecke algebra relations (2.2). Decomposing $\Psi = \sum_\pi \Psi_\pi |\pi\rangle$ in the path representation basis leads to

$$t_i \Psi_\pi = \sum_{\substack{\pi' \neq \pi \\ \pi \in e_i \pi'}} \Psi_{\pi'}, \qquad i = 1, 2, \ldots, N-1, \tag{3.3}$$

where the notation $\pi \in e_i\pi'$ simply means that we select π' such that $C_{i,\pi',\pi} = 1$.

4. Minimal polynomial solutions

Looking for polynomial solutions of minimal degree to the set of equations (3.1), we have found that

$$s = q^{2(k+1)} \qquad \text{and} \qquad c = ((-1)^k q^{k+1})^{n-1}, \tag{4.1}$$

so that these are solutions at level 1 and that

$$\Psi_{\pi_0} = \prod_{m=1}^{k} \prod_{1+(m-1)n \leqslant i < j \leqslant mn} (qz_i - q^{-1}z_j). \tag{4.2}$$

To see why, first consider equations (3.3) and set $z_{i+1} = q^2 z_i$: this implies $e_i\Psi = \tau\Psi$, hence if $\tau \neq 0$ (i.e. $q^2 \neq -1$), then all the components Ψ_π where π is not in the image of some π' under e_i must vanish. Thanks to property (P3), we see that

$$\Psi_\pi|_{z_{i+1}=q^2 z_i} = 0 \qquad \text{if} \quad \pi_i \geqslant \pi_{i+1}. \tag{4.3}$$

By appropriate iterations, this is easily extended to concave portions of π, namely such that $\pi_i \geqslant \pi_{i+1} \geqslant \cdots \geqslant \pi_{i+j}$, for which Ψ_π vanishes at $z_l = q^2 z_k$, for any pair k, l such that

[3] The more standard generator $T_i = q(q - t_i)$ of the Hecke algebra (which satisfies the braid relations) is nothing but the Lusztig operator [9].

$i \leqslant k < l \leqslant j$. Henceforth, as π_0 has k flat portions $\pi_{(m-1)n+1} = \pi_{(m-1)n+2} = \cdots = \pi_{mn} = m, m = 1, 2, \ldots, k$, separated by convex points we find that Ψ_{π_0} must factor out expression (4.2), which is the minimal realization of this property. Alternatively, this may be recast into the highest weight condition that $(t_i + \tau)\Psi_{\pi_0} = 0$ for all i not multiple of n, and (4.2) is the polynomial of smallest degree satisfying it. Note that once Ψ_{π_0} is fixed, all the other components of Ψ are determined by equations (3.3), and therefore the cyclicity condition (3.1*b*) is automatically satisfied, with the values of c_N and s (4.1) fixed by compatibility. This is a consequence of the property (P4), which allows us to express any Ψ_π with $\pi_i < \pi_{i+1}$ only in terms of $\Psi_{\pi'}$s such that $\pi \subset \pi'$ in the above sense, henceforth in a triangular way w.r.t. inclusion of paths.

Another consequence of this property is that if we pick say $z_m = z, z_{m+1} = q^2 z, \ldots, z_{m+k-1} = q^{2(k-1)} z$ for $m + k \leqslant N$, then the only possibly non-vanishing components Ψ_π of Ψ are those having the convex sequence $\pi_m = 1, \pi_{m+1} = 2, \ldots, \pi_{m+k-1} = k$. Let $\varphi_{m,m+k-1}$ denote the embedding of $SU(k)$ paths of length $(n-1)k$ into those of length nk obtained by inserting a convex sequence of k steps $1, 2, \ldots, k$ between the $(m-1)$th and mth steps. Then we have the following recursion relation:

$$\Psi_{\varphi_{m,m+k-1}(\pi)}(z_1, \ldots, z_N)\big|_{z_{m+k-1}=q^2 z_{m+k-2}=\cdots=q^{2(k-1)}z_m}$$
$$= C\left(\prod_{i=1}^{m-1}(qz_i - q^{-1}z_m)\prod_{i=m+k}^{N}(qz_{m+k-1} - q^{-1}z_i)\right)\Psi_\pi(z_1, \ldots, z_{m-1}, z_{m+k}, \ldots, z_N) \tag{4.4}$$

for some constant C. This is readily obtained by first noting that equations (3.3) are still satisfied by the lhs of (4.4) for $i = 1, 2, \ldots, m-2$ and $i = m+k, m+k+1, \ldots, N$, while the prefactor on the rhs remains unchanged. Moreover, expressing the interchange of z_{m-1} and z_{m+k} on the lhs as suitable successive actions of $\check{R}$ matrices yields the missing equation.

An alternative characterization of this polynomial solution is that all components of Ψ vanish whenever we restrict to

$$z_{i_1} = z, \quad z_{i_2} = q^2 z, \quad \ldots, \quad z_{i_{k+1}} = q^{2k} z \tag{4.5}$$

for some ordered $(k+1)$uple $i_1 < i_2 < \cdots < i_{k+1}$. This generalizes the observation of Pasquier [6] to the $SU(k)$ case and allows us to interpret our solution as some q-deformed quantum Hall effect wavefunctions, with filling fraction $\nu = k$, bound to vanish whenever $k+1$ particles come into contact, up to shifts of q^{2j}, the so-called 'quantum incompressibility' condition.

Note that level 1 highest weight representations of $U_q(\widehat{\mathfrak{sl}(k)})$ have been extensively studied in the literature; in particular, q-bosonization techniques lead to integral formulae for solutions of level 1 qKZ equation, see e.g. [10], but they are usually expressed in the spin basis.

5. Generalized Razumov–Stroganov sum rules and generalized ASM numbers

To derive a sum rule for the components of Ψ, we introduce a covector v such that

$$ve_i = \tau v, \quad i = 1, 2, \ldots, N-1, \qquad \text{and} \qquad v\sigma = v. \tag{5.1}$$

Expressing $vY_k(e_i, \ldots, e_{i+k-1}) = 0$, we find that $U_1(\tau)U_2(\tau)\cdots U_k(\tau) = 0$ and further demanding that v have positive entries fixes

$$q = -e^{\frac{i\pi}{k+1}}, \qquad \text{i.e.} \quad \tau = 2\cos\left(\frac{\pi}{k+1}\right). \tag{5.2}$$

Note that equations (5.1) and the above path representation allow for writing a manifestly positive formula for the entries of v only in terms of Chebyshev polynomials, and we choose the normalization $v_{\pi_f} = 1$. For illustration, for $k = 3, n = 2$, v is indexed by the paths (2.3), and we have $v_{\pi_f} = 1$, $v_{\pi_3} = U_1 = \sqrt{2}$, $v_{\pi_2} = v_{\pi_1} = U_2 = 1$ and $v_{\pi_0} = U_1 U_2 = \sqrt{2}$. The covector v clearly satisfies $v\check{R}_{i,i+1} = v$ for $i = 1, 2, \ldots, N-1$ from the explicit form (2.1). We deduce that the quantity $v \cdot \Psi$ is invariant under the action of τ_i, $i = 1, 2, \ldots, N-1$, as a direct consequence of (3.1*a*) and of the first line of (5.1), hence is fully symmetric in z_is. This symmetry is compatible with the cyclic relation (3.1*b*) as $c = s = 1$ from (5.2). Note that in the spin basis, $v \cdot \Psi$ is nothing but the sum of all components of Ψ.

An important remark is in order: as we have $c = s = 1$, Ψ is actually the suitably normalized ground-state vector of the fully inhomogeneous A_{k-1} IRF model on a semi-infinite cylinder of perimeter N, defined via its (periodic) transfer matrix acting on the path basis states. Indeed, this transfer matrix is readily seen to be (i) intertwined by the matrices $\check{R}_{i,i+1}(z_{i+1}, z_i)$ and (ii) cyclically symmetric under a rotation by one step along the boundary of the half-cylinder, hence if Ψ denotes the (Perron–Frobenius) ground-state eigenvector of this transfer matrix, then equations (3.1) follow, with $c = s = 1$. These models reduce to the half-cylinder $O(1)$ loop model for $k = 2$, upon identifying the 'Dyck path' basis with that of link patterns. In that case, the covector v is simply $(1, 1, \ldots, 1)$ and gives rise to the Razumov–Stroganov sum rule for $v \cdot \Psi$, proved in [3]. For general k, the corresponding sum rule reads

$$i^{kn(n-1)/2} v \cdot \Psi = s_Y(z_1, \ldots, z_N), \tag{5.3}$$

where $i = \sqrt{-1}$, s is a Schur function and Y is the Young diagram with k rows of $(n-1)$ boxes, k rows of $(n-2)$ boxes, $\ldots$, k rows of 1 box. Equation (5.3) is proved as follows: by construction, $v \cdot \Psi$ is a symmetric polynomial of total degree $kn(n-1)/2$ and partial $n-1$ in each z_i, which moreover satisfies recursion relations inherited from (4.4), upon noting that $v_{\varphi_{m,m+k-1}(\pi)} \big/ v_\pi$ is independent of the path π. The Schur function on the rhs is the unique symmetric polynomial of zs with the correct degree and subject to these recursion relations (or alternatively to the vanishing condition (4.5)). The remaining global normalization is fixed by induction, by comparing both sides at $n = 1$.

Using the explicit definition of Schur functions, we get the following homogeneous limits, when all z_is tend to 1:

$$i^{kn(n-1)/2} v \cdot \Psi_{\text{Hom}} = s_Y(1, 1, \ldots, 1) = (k+1)^{n(n-1)/2} \frac{\prod_{i=1}^{k-1} \prod_{j=0}^{n-1} ((k+1)j + i)!}{\prod_{j=0}^{n(k-1)-1} (n+j)!}, \tag{5.4}$$

where the quantities

$$A_n^{(k)} = \frac{\prod_{i=1}^{k-1} \prod_{j=0}^{n-1} ((k+1)j + i)!}{\prod_{j=0}^{n(k-1)-1} (n+j)!} \tag{5.5}$$

are integers that generalize the numbers of alternating sign matrices of size $n \times n$, recovered for $k = 2$. Note also that $A_2^{(k)} = c_k$, the kth Catalan number.

It would be extremely interesting to find a combinatorial interpretation for these integer numbers, possibly in terms of 'domain wall boundary condition' partition functions of the associated $SU(k)$ vertex model. Note however that as opposed to the $SU(2)$ case, the components of Ψ are no longer integers.

6. Rational limit and geometry: extended Joseph polynomials

For generic q, one could hope the polynomials defined above to be geometrically interpreted in terms of K-theory; this is however beyond the scope of the present letter, and we only consider here the rational limit, which is obtained by substituting

$$q = -e^{-\epsilon a/2}, \qquad z_i = e^{-\epsilon w_i}, \qquad i = 1, 2, \ldots, N, \tag{6.1}$$

and expanding to first non-trivial order in ϵ as it goes to zero.

The qKZ equation becomes the 'rational qKZ equation': it has the same form as before, but the R-matrix is now the rational solution of YBE related to quotients of the symmetric group $\mathcal{S}_N$.

In this limit, the qKZ polynomial solutions at level 1 remain polynomials in $a, w_1, \ldots, w_N$. They turn out to have a remarkable algebro-geometric interpretation, in the same spirit as [5]: they are 'extended' Joseph polynomials [11] associated with the Young diagram of rectangular shape $k \times n$. More precisely, consider the scheme of nilpotent $N \times N$ complex upper triangular matrices U which satisfy $U^k = 0$. It is well known that its irreducible components are indexed by Standard Young Tableaux (SYT) of rectangular shape $k \times n$; the latter are equivalent to paths in the Weyl chamber of $SU(k)$, according to the following rule: the numbers on the ith row of the SYT record the positions of steps of type i of the path. With each path one can then associate the equivariant multiplicity [12, 13] (also called *multidegree*) of the corresponding component with respect to the action of the torus $(\mathbb{C}^\times)^{N+1}$, where an N-dimensional torus acts by conjugation of U by diagonal matrices and the extra one-dimensional torus acts by an overall rescaling of U. This results in a set of homogeneous polynomials in $N + 1$ variables $a, w_1, \ldots, w_N$, which turn out to coincide with the entries of Ψ in the limit (6.1). We call these polynomials *extended Joseph polynomials* because the usual Joseph polynomials [13], defined without the rescaling action, correspond to $a = 0$. On the other hand, note that setting $w_i = 0$ and $a = 1$ yields the degrees of the components.

The fact that extended Joseph polynomials satisfy the limit (6.1) of equations (3.3) is nothing but Hotta's explicit construction [14] of the Joseph/Springer representation. This will be discussed in detail elsewhere [11]. As a corollary, note that at $a = 0$, the (usual) Joseph polynomials satisfy the (usual) level 1 Knizhnik–Zamolodchikov equation [15], which takes the form

$$(k+1)\frac{\partial}{\partial w_i}\Psi = \sum_{j(\neq i)} \frac{s_{i,j}+1}{w_i - w_j}\Psi, \tag{6.2}$$

where $s_{i,j}$ is the transposition (ij) in $\mathcal{S}_N$ (explicitly, for $i < j, s_{j,i} = s_{i,j} = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i$ with $s_i = 1 - e_i$). The solutions of these equations are well known in the spin basis, indexed by sequences $(\alpha_1, \ldots, \alpha_N)$ in which every number from 1 to k occurs n times. They are simple products of Vandermonde determinants: $\Psi_{\alpha_1,\ldots,\alpha_N}(0, w_1, \ldots, w_N) = \prod_{\beta=1}^{k} \prod_{i,j\in A_\beta, i<j}(w_i - w_j)$, where $A_\beta = \{i : \alpha_i = \beta\}$.

A last remark is in order. Throughout this letter, we have restricted ourselves for simplicity to systems of size $N = kn$ multiple of k. However, we may obtain solutions of the qKZ equation for arbitrary size say $N = kn - j$ from that of size $N = nk$ by letting successively $z_{kn} \to 0, z_{kn-1} \to 0, \ldots, z_{kn-j+1} \to 0$ in Ψ. This immediately yields other sum rules of the form (5.3), but with a Schur function for a truncated Young diagram, with its j first rows deleted. Similarly, we have access to the multidegrees of the scheme of nilpotent upper triangular matrices of arbitrary size as well.

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